RESEARCH PAPERS

Theoretical model of particle orientation distribution function in a cylindrical particle suspension subject to turbulent shear flow*

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Abstract The equation for the orientation probability function of slender cylindrical particles suspended in planar turbulent flows was investigated. After ensemble averaging, the equations for the mean and fluctuating probability function were derived. The equation for the fluctuating probability function appearing in the equation of mean probability function was solved by using the method of characteristics analysis. The orientational dispersion terms due to the random motion of cylindrical particles in the equation of mean probability function are related to the mean probability function and the Lagrangian fluid velocity correlations. The evolution of the mean probability function was described by a modified orientation-space-convection equation, where the dispersion terms account for the randomizing effect of the turbulence.

Keywords: **dispersion, cylindrical particle suspensions, turbulent flow, probability function.**

Understanding the microstructure of slender cylindrical particle suspensions subject to turbulent flows is currently of great importance in both theoretical investigation and practical applications. The practical importance arises from many industrial processes, such as pulp and paper industry, where nearly all fiber processing and papermaking is performed at high speeds in turbulent fluids. It is also of theoretical importance in understanding how to model fluid — particle interactions in non-spherical particle suspensions in turbulent flows.

Jeffery[1] has solved the motion of a single non-spherical particle in an unbounded linear shear flow. He found that the particle rotates in one of a family of periodic and close orbits around the vorticity axis. However, external actions, such as particle-particle interactions or fluid random motion, can cause small disturbances of the particles. Several authors (Shaqfeh & Fredrickson[2], Koch[3], Petrich et al.[4], Rahnama et al.[5], and Shi & Lin[6]) have investigated the particle-particle interactions in simple planar flows. However, there are few investigations on non-spherical particle suspensions subject to turbulent flows because of the heavy computational requirements in modeling the translations and rotations of a great number of particles. Cho et al.[7] provided one of the first investigations of the effect of turbulence on the orientation of high aspect ratio ice crystals. Krushkal & Gallily[8] theoretically calculated the orientation distribution function of small fibers in turbulent flow by using the Fokker-Planck equation. Recently, Olson & Kerekes[9] obtained the fiber translational and rotational dispersion coefficients with the assumption that the relative velocity of particle and fluid can be neglected. Lin et al.[10] analyzed the behavior of fiber suspension and fluid stress in an evolving mixing layer of fiber suspensions. You et al.[11] studied the stability in the channel flow of fiber suspensions with spectral method. Lin et al.[12,13] studied the orientation distribution of fibers in a mixing layer and a turbulent pipe flow, respectively. Gao et al.[14] derived the translational and rotational dispersion coefficients of fiber taking the balance between the Stokes drag and virtual mass force into consideration, and discussed the dispersion property of fibers, long and short.

All these investigations mentioned above, however, are performed based on the Lagrangian ap-

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proach, which suffers from large computational requirements. Therefore, the purpose of this paper is to give a model to describe the cylindrical particle orientations in planar inhomogeneous turbulent flows based on the Eulerian approach. Olson & Kerekes[9] have derived the orientational dispersion coefficients for fibers in the homogeneous turbulent flows. In their model, the turbulent dispersions are independent of the particle orientations due to the homogeneous turbulence. However, in the inhomogeneous flows, the turbulent dispersion coefficients are not the same at different particle orientations. Therefore, in this paper, we will derive the equation for the orientation probability function of cylindrical particles suspended in turbulent planar flows, then give the turbulent dispersion coefficients by using the derived equation, from which we can make predictions of the state of the cylindrical particles during flows and also further investigate the effect of the existence of particles on turbulent flows.

1 The equation of the probability function

We consider a suspension with n rigid, cylindrical particles per unit volume; each particle has the length of L and diameter of D. The particle aspect ratio, L/D, is assumed very large so that the end effect of particles can be neglected. The particle concentration considered in this paper is $nL^3 \ll 1$, which means the suspension is dilute.

An important step in the present model is to simplify the expression for the particle motion. To accomplish this goal we make the following assumptions:

(i) Particles are regarded as moving with the fluid particle at their centers, that is, the centroids of the particles move like points. Particles are small enough so that the fluid velocity gradients are assumed homogeneous within the particle lengths. According to Jeffery's result[1] for the particle with infinite aspect ratio, the time rate of change of the particle orientation is given by

$$\dot{p}_i = u_{i,j}p_j - u_{j,l}p_l p_j p_i, \tag{1}$$

where $u_{i,j} = \partial u_i / \partial x_j$ is the fluid velocity gradient, p_i is a component of unit vector parallel to the particle axis (Fig. 1):

$$\begin{cases} p_1 = \sin\theta\cos\phi, \\ p_2 = \sin\theta\sin\phi, \\ p_3 = \cos\theta. \end{cases} \tag{2}$$

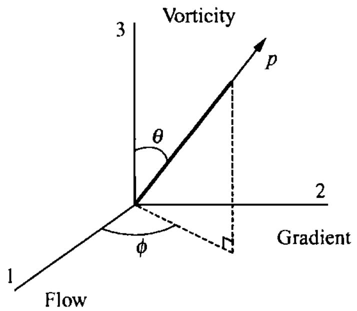

Fig. 1. A cylindrical particle in the spherical coordinate system.

(ii) A distribution function $\psi(x_i, p_j, t)$ is introduced to represent the probability that the test particle selected has a specific location x_i and orientation p_j at time t. The distribution function ψ can be assumed as:

$$\psi(x_i, p_j, t) = n(x_i, t)\varphi(x_i, p_j, t), \tag{3}$$

where n is a position-dependent number density. We further assume that there are no concentration gradients so that n is a constant. To be consistent with the definition of ψ, φ is normalized:

$$\int \varphi(x_i, p_j, t)\mathrm{d}p_j = 1. \tag{4}$$

The governing equation for φ depends on the conservation of cylindrical particles in the orientation space. The transient equation for particle probability distribution function is given by

$$\frac{\partial\varphi}{\partial t} + \frac{\partial(\dot{p}_i\varphi)}{\partial p_i} = 0. \tag{5}$$

For planar flows, we write the velocity gradients as

$$u_{1,1} = -u_{2,2} = \varepsilon_1, \quad u_{1,2} = \varepsilon_2, \quad u_{2,1} = \varepsilon_3. \tag{6}$$

In the spherical coordinate system as shown in Fig. 1, the steady state equation for the orientation probability function has the following form:

$$\dot{\theta}\frac{\partial\varphi}{\partial\theta} + \dot{\phi}\frac{\partial\varphi}{\partial\phi} + s\varphi = 0, \tag{7}$$

where

$$\begin{aligned} \dot{\theta} &= \sin\theta\cos\theta \times (\cos 2\phi\varepsilon_1 + \sin\phi\cos\phi\varepsilon_2 + \sin\phi\cos\phi\varepsilon_3) \\ &= \sin\theta\cos\theta \cdot f(\phi, \varepsilon_1, \varepsilon_2, \varepsilon_3), \end{aligned} \tag{8}$$

$$\begin{aligned} \dot{\phi} &= -2\sin\phi\cos\phi\varepsilon_1 - \sin^2\phi\varepsilon_2 + \cos^2\phi\varepsilon_3 \\ &= g(\phi, \varepsilon_1, \varepsilon_2, \varepsilon_3), \end{aligned} \tag{9}$$

$$s = -3\sin^2\theta \cdot f(\phi, \varepsilon_1, \varepsilon_2, \varepsilon_3). \tag{10}$$

Using the method of characteristics, Eq. (7) can be changed into three simple differential equations:

$$\frac{d\theta}{d\tau} = \dot{\theta} = \sin\theta\cos\theta f(\phi), \tag{11}$$

$$\frac{d\phi}{d\tau} = \dot{\phi} = g(\phi), \tag{12}$$

$$\frac{d\varphi}{d\tau} + s\varphi = 0. \tag{13}$$

The boundary condition is

$$\varphi(\theta = \sigma, \phi = \pi/2, \tau = 0) = c(\text{const}). \tag{14}$$

Then we have

$$\varphi = \frac{k(\sigma)}{\cos^3\theta} = c\frac{\cos^3\sigma}{\cos^3\theta}, \tag{15}$$

where k is an orbit constant, which accounts for the different characteristic; c is a constant determined by the normalization condition (4). From Eqs. (11) and (12) we have

$$\tau = \int_{\pi/2}^{\phi} g^{-1}(\phi_0)d\phi_0, \tag{16}$$

$$\tan\sigma = \tan\theta \cdot \exp\left(-\int_{\pi/2}^{\phi} f(\phi_0) \cdot g^{-1}(\phi_0)d\phi_0\right). \tag{17}$$

Substituting Eq. (17) into Eq. (15), we obtain the final expression for the orientation probability function

$$\varphi = c\left[\cos^2\theta + \sin^2\theta \cdot \exp\left(-2\int_{\pi/2}^{\phi} f(\phi_0) \cdot g^{-1}(\phi_0)d\phi_0\right)\right]^{-\frac{3}{2}}. \tag{18}$$

2 Solution of the fluctuating equation

As a description in the Eulerian field, the probability function is the statistical quantity in the orientation space. Given a known flow field, the probability function at a certain position can be determined by the local transient velocity gradients. To estimate the random effect of turbulence on the behavior of the cylindrical particles, the fluid velocity field should be averaged by using the ensemble average method (or volume average method in the position space). For turbulent suspensions, the particles undergo mean motion due to the mean fluid velocity and random motion due to the fluctuating component of the fluid velocity. Thus, the probability function φ can be expressed as:

$$\varphi = \bar{\varphi} + \varphi', \tag{19}$$

where $\bar{\varphi}$ is the mean probability function, which means the volume average of φ in the position space, and φ' is the fluctuating probability function. Averaging Eq. (7), we obtain the equation for $\bar{\varphi}$

$$\bar{\dot{\theta}}\frac{\partial\bar{\varphi}}{\partial\theta} + \bar{\dot{\phi}}\frac{\partial\bar{\varphi}}{\partial\phi} + \bar{s}\,\bar{\varphi} = \bar{P}, \tag{20}$$

and then get the equation for φ'

$$\bar{\dot{\theta}}\frac{\partial\varphi'}{\partial\theta} + \bar{\dot{\phi}}\frac{\partial\varphi'}{\partial\phi} + \bar{s}\varphi' = Q, \tag{21}$$

where

$$\bar{P} = -\overline{\dot{\theta}'\frac{\partial\varphi'}{\partial\theta} - \dot{\phi}'\frac{\partial\varphi'}{\partial\phi} - s'\varphi'}, \tag{22}$$

$$Q = -\dot{\theta}'\frac{\partial\bar{\varphi}}{\partial\theta} - \dot{\phi}'\frac{\partial\bar{\varphi}}{\partial\phi} - s'\bar{\varphi}. \tag{23}$$

Equation (20) is the key equation to understand the mean motion of the cylindrical particles suspended in turbulent flows. The expression for $\bar{P}$ in Eq. (23) includes several correlation terms, which require the information of the evolution of φ'. If we assume that

$$\bar{P} = 0, \tag{24}$$

the solution of Eq. (20) is similar to the expression of (15):

$$\bar{\varphi} = c \cdot \frac{\cos^3\sigma}{\cos^3\theta}, \tag{25}$$

where

$$\tan\sigma = \tan\theta \cdot \exp\left(-\int_{\pi/2}^{\phi} \bar{f}(\phi_0) \cdot \bar{g}^{-1}(\phi_0)d\phi_0\right). \tag{26}$$

Using the method of characteristics again, the equation for φ' in Eq. (21) can be changed into three differential equations:

$$\frac{d\theta}{d\tau} = \bar{\dot{\theta}} = \sin\theta\cos\theta\bar{f}(\phi), \tag{27}$$

$$\frac{d\phi}{d\tau} = \bar{\dot{\phi}} = \bar{g}(\phi), \tag{28}$$

$$\frac{d\varphi'}{d\tau} + \bar{s}\varphi' = Q. \tag{29}$$

The boundary condition is

$$\varphi'(\theta = \sigma, \phi = \pi/2, \tau = 0) = 0. \tag{30}$$

Solving Eq. (29), we get

$$\varphi' = \frac{\int_0^{\tau} h(\tau_0, \sigma)d\tau_0}{\cos^3\theta}, \tag{31}$$

where

$$h(\tau, \sigma) = \cos^3\theta \cdot Q. \tag{32}$$

At the same time, the expressions for τ and σ are

$$\tau = \int_{\pi/2}^{\phi} \bar{g}^{-1}(\phi_0)d\phi_0, \tag{33}$$

and

$$\tan\sigma = \tan\theta \cdot \exp\left(- \int_{\pi/2}^{\phi} \bar{f}(\phi_0) \cdot \bar{g}^{-1}(\phi_0) \mathrm{d}\phi_0 \right), \tag{34}$$

respectively. It should be noted that Eq. (34) and Eq. (26) have the same forms.

3 The expression for the dispersion terms

It is a key problem to analyze the expression of $\bar{P}$, the expression includes several correlation terms involving the information of φ'. Based on the result of φ', we can relate $\bar{P}$ to the mean probability function, and the correlation terms can be changed into Lagrangian fluid velocity correlations.

Considering Eq. (21), we have

$$\frac{\partial \varphi'}{\partial \theta} - 3\tan\theta\varphi' = \frac{1}{\sin\theta\cos\theta \cdot \bar{f}(\phi)} \left(Q - \bar{g}(\phi) \frac{\partial \varphi'}{\partial \phi} \right). \tag{35}$$

Substituting Eq. (35) into Eq. (22), the expression of P can be written as

$$- P = \frac{f'}{\bar{f}} Q + \left(g' - \frac{\bar{g}}{\bar{f}} f' \right) \frac{\partial \varphi'}{\partial \phi}. \tag{36}$$

From Eq. (31), $\partial\varphi'/\partial\phi$ is given by

$$\frac{\partial \varphi'}{\partial \phi} = \frac{\left(h(\tau, \sigma) \dfrac{\partial \tau}{\partial \phi} + \displaystyle\int_0^{\tau} \frac{\partial h(\tau_0, \sigma)}{\partial \sigma} \mathrm{d}\tau_0 \cdot \frac{\partial \sigma}{\partial \phi} \right)}{\cos^3\theta} = \frac{\left(Q - \dfrac{\sin\sigma\cos\sigma \cdot \bar{f}}{\cos^3\theta} \displaystyle\int_0^{\tau} \frac{\partial h(\tau_0, \sigma)}{\partial \sigma} \mathrm{d}\tau_0 \right)}{\bar{g}}. \tag{37}$$

Letting

$$\lambda = \frac{\sin\sigma\cos\sigma \cdot \bar{f}}{\cos^3\theta \cdot Q} \int_0^{\tau} \frac{\partial h(\tau_0, \sigma)}{\partial \sigma} \mathrm{d}\tau_0, \tag{38}$$

we have

$$\frac{\partial \varphi'}{\partial \phi} = \frac{(1 - \lambda) Q}{\bar{g}}, \tag{39}$$

and

$$- P = Q \cdot \left[\frac{g'}{\bar{g}} + \lambda \left(\frac{f'}{\bar{f}} - \frac{g'}{\bar{g}} \right) \right]. \tag{40}$$

The integral in Eq. (38) involves the mean probability function along the orbits. Since the mean probability function is unknown, we assume that the expression of λ is determined by Eq. (25), which is the solution of the equation for the mean probability function neglecting the dispersion terms.

Considering Eq. (24), the expression for $\cos^3\theta \cdot Q$ can be written as

$$\cos^3\theta \cdot Q = \cos^3\theta \cdot \left(\frac{\bar{g}}{\bar{f}} f' - g' \right) \frac{\partial \bar{\varphi}}{\partial \phi} = \left(\frac{\bar{g}}{\bar{f}} f' - g' \right) \frac{\partial (\cos^3\theta \cdot \bar{\varphi})}{\partial \phi}. \tag{41}$$

Substituting Eq. (25) into Eq. (41) we have

$$\cos^3\theta \cdot Q = 3c \cdot \left(f' - \frac{\bar{f}}{\bar{g}} g' \right) \sin^2\sigma\cos^3\sigma = H(\phi, \sigma). \tag{42}$$

According to Eq. (33), the integral in Eq. (38) can be changed into the integral along the angle ϕ. The new expression of λ is

$$\lambda = \frac{5\cos^2\sigma - 3}{\left(\dfrac{f'}{\bar{f}} - \dfrac{g'}{\bar{g}} \right)} \int_{\pi/2}^{\phi} \frac{1}{\bar{g}} \cdot \left(f' - \frac{\bar{f}}{\bar{g}} g' \right) \mathrm{d}\phi_0, \tag{43}$$

where the expression for σ is determined by Eq. (34).

Combined with the above equation, Eq. (40) gives the final expression of the correlation terms. To simplify the expressions, we let

$$a_1 = \cos 2\phi, \quad a_2 = \sin\phi\cos\phi, \quad a_3 = \sin\phi\cos\phi, \tag{44}$$

$$b_1 = - 2\sin\phi\cos\phi, \quad b_2 = - \sin^2\phi, \quad b_3 = \cos^2\phi, \tag{45}$$

$$\begin{cases} c_1 = \bar{g} \cdot (5\cos^2\sigma - 3) \displaystyle\int_{\pi/2}^{\phi} \frac{1}{\bar{g}} \cdot \left(a_1 - \frac{\bar{f}}{\bar{g}} b_1 \right) \mathrm{d}\phi_0, \\ c_2 = \bar{g} \cdot (5\cos^2\sigma - 3) \displaystyle\int_{\pi/2}^{\phi} \frac{1}{\bar{g}} \cdot \left(a_2 - \frac{\bar{f}}{\bar{g}} b_2 \right) \mathrm{d}\phi_0, \\ c_3 = \bar{g} \cdot (5\cos^2\sigma - 3) \displaystyle\int_{\pi/2}^{\phi} \frac{1}{\bar{g}} \cdot \left(a_3 - \frac{\bar{f}}{\bar{g}} b_3 \right) \mathrm{d}\phi_0. \end{cases} \tag{46}$$

Thus we have

$$f' = \sum_i a_i \varepsilon_i', \quad g' = \sum_i b_i \varepsilon_i', \quad \lambda \left(\frac{f'}{\bar{f}} - \frac{g'}{\bar{g}} \right) = \frac{1}{\bar{g}} \sum_i c_i \varepsilon_i'. \tag{47}$$

The expression of $\bar{P}$ is in the following form:

$$\bar{P} = \frac{\sin\theta\cos\theta}{\bar{g}} \cdot \sum_{i,j} [a_i(b_j + c_j) \cdot \overline{\varepsilon_i' \varepsilon_j'}] \cdot \frac{\partial \bar{\varphi}}{\partial \theta} + \frac{1}{\bar{g}} \cdot \sum_{i,j} [b_i(b_j + c_j) \cdot \overline{\varepsilon_i' \varepsilon_j'}] \cdot \frac{\partial \bar{\varphi}}{\partial \phi} - \frac{3\sin^2\theta}{\bar{g}} \cdot \sum_{i,j} [a_i(b_j + c_j) \cdot \overline{\varepsilon_i' \varepsilon_j'}] \cdot \bar{\varphi}. \tag{48}$$

From the above equations, we can relate the correlation terms to the mean probability function and its gradients in the orientation space. According to the definition (6), $\overline{\varepsilon_i'\varepsilon_j'}$ is the Lagrangian fluid velocity correlation. Combined with Eq. (20), Eq. (48) provides a bridge to understand the influence of turbulent fluids on the orientation distribution of cylindrical particles suspended in the flows.

The expressions of the coefficients c_i are not in the apparent form because of the involved integrals along the angle ϕ. These integrals can be calculated numerically for general planar flows. However, for simple planar shear flows, these coefficients can be given directly.

For shear flows, the mean fluid motion is defined as:

$$\bar{\varepsilon}_1 = \bar{\varepsilon}_3 = 0. \tag{49}$$

The expressions for $\bar{f}$ and $\bar{g}$ are

$$\begin{cases} \bar{f} = \sin\phi\cos\phi\bar{\varepsilon}_2, \\ \bar{g} = -\sin^2\phi\bar{\varepsilon}_2. \end{cases} \tag{50}$$

Substituting Eqs. (49)—(50) into Eq. (34), we have

$$\tan\sigma = \tan\theta\sin\phi. \tag{51}$$

At the same time, the coefficients c_i can be calculated directly by

$$\begin{cases} c_1 = \sin\phi\cos\phi \cdot \left(\dfrac{5}{1+\tan^2\theta\sin^2\phi} - 3\right), \\ c_2 = 0, \\ c_3 = -\dfrac{1}{2}\cos^2\phi \cdot \left(\dfrac{5}{1+\tan^2\theta\sin^2\phi} - 3\right). \end{cases} \tag{52}$$

4 Conclusion

The self-governed equation for mean probability function has been proposed. Two key steps are adopted in obtaining this equation. One step is to solve the equation for the fluctuating probability function by using the method of characteristics. The solution of the fluctuating equation includes an integral along the particle orbits described by Eq. (33). To avoid the difficult problem of considering the coupled mean and fluctuating equations, the second key step is to determine the dispersion coefficients by using the solution of the simplified mean equation for the probability function. In this way, the dispersion terms, which represent the fluctuating effect of turbulence, are related to Lagrangian fluid velocity correlations.

References

1 Jeffery G. B. Motion of ellipsoidal particles immersed in a viscous fluid. Proc. R. Soc. London Ser. A, 1922, 102: 161—179.
2 Shaqfeh E. S. G. and Fredrickson G. H. The hydrodynamic stress in a suspension of rods. Phys. Fluids A, 1990, 2: 7—24.
3 Koch D. L. A model for orientational diffusion in fiber suspensions. Phys. Fluids, 1995, 7: 2086—2088.
4 Petrich M. P., Koch D. L. and Cohen C. An experimental determination of the stress-microstructure relationship in semi-concentrated fiber suspensions. J. Non-Newtonian Fluid Mech., 2000, 95: 101—133.
5 Rahnama M. D., Koch D. L. and Shaqfeh E. S. G. The effect of hydrodynamic interactions on the orientation distribution in a fiber suspension subject to simple shear flow. Phys. Fluids, 1995, 7: 487—501.
6 Shi X. and Lin J. Z. Direct simulation on two-phase flows with two-way coupling velocity model. Progress in Natural Science, 2001, 11(2): 149—155.
7 Cho H. E., Iribane J. and Richard W. On the orientation of ice crystals in cumulonimbus cloud. J. Atmos. Sci., 1981, 38: 111—113.
8 Krushkal E. and Gallily I. On the orientation distribution function of non-spherical aerosol particles in a general shear flow-II. The turbulent Case. Journal of Aerosol Science, 1988, 19: 197—211.
9 Olson J. A. and Kerekes R. J. The motion of fibers in turbulent flow. J. Fluid Mech., 1998, 377: 47—64.
10 Lin J. Z., Shi X., Shao X. M. et al. Research on the orientation of fiber and fluid stress in a mixing layer of fiber suspensions. Progress in Natural Science (in Chinese), 2002, 12(4): 372—376.
11 Lin J. Z. and You Z. J. Stability in channel flow with fiber suspensions. Progress in Natural Science, 2003, 11(2): 95—99.
12 Lin J. Z., Shi X. and Yu Z. S. The motion of cylindrical particles in an evolving mixing layer. Int. J. of Multiphase Flow, 2003, 29(8): 1355—1372.
13 Lin J. Z., Zhang W. F. and Yu Z. S. Numerical research on the orientation distribution of fibers immersed in laminar and turbulent pipe flows. Journal of Aerosol Science, 2004, 35(1): 63—82.
14 Gao Z. Y. and Lin J. Z. Research on the dispersion of cylindrical particles in turbulent flows. Journal of Hydrodynamics, 2004, 16(1): 1—6.